

FINITE RANK ISOPAIRS

UDENI WIJESOORIYA

ABSTRACT. An algebraic isopair is a commuting pair of pure isometries that is annihilated by a polynomial defining a distinguished variety \mathcal{V} . The notion of the rank of a pure algebraic isopair with finite bimultiplicity is introduced. For \mathcal{V} , a union of s irreducible varieties \mathcal{V}_j , the rank is a s -tuple $\alpha = (\alpha_1, \dots, \alpha_s)$ of natural numbers. A pure algebraic isopair of finite bimultiplicity with rank α is described as a restriction of a $\max\{\alpha_1, \dots, \alpha_s\}$ -cyclic pure algebraic isopair to a finite codimensional invariant subspace. The restriction of a pure algebraic isopair of finite bimultiplicity with rank α to a finite codimensional invariant subspace is at least $\max\{\alpha_1, \dots, \alpha_s\}$ -cyclic and there is a $\max\{\alpha_1, \dots, \alpha_s\}$ -cyclic finite codimensional invariant subspace.

1. INTRODUCTION

Given a polynomial $p \in \mathbb{C}[z, w]$ (or in $\mathbb{C}[z]$) let $Z(p)$ denote its zero set. We say p is *square free* if q^2 does not divide p for every non-constant polynomial $q(z, w) \in \mathbb{C}[z, w]$. We say $q \in \mathbb{C}[z, w]$ is the *square free version* of p if q is the polynomial with smallest degree such that q divides p and $Z(p) = Z(q)$. The square free version is unique up to multiplication by a nonzero constant.

Let \mathbb{D} , \mathbb{T} and \mathbb{E} denote the open unit disk, the boundary of the unit disk and complement of the closed unit disk in \mathbb{C} respectively. In [AKM12] the notion of an inner toral polynomial is introduced. (See also [AMS06, AM06, K10, PS14].) A polynomial $q \in \mathbb{C}[z, w]$ is *inner toral* if

$$Z(q) \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2.$$

In other words, if $(z, w) \in Z(q)$ then either $|z|, |w| < 1$ or $|z| = 1 = |w|$ or $|z|, |w| > 1$. A *distinguished variety* in \mathbb{C}^2 is the zero set of an inner toral polynomial.

Let V be an isometry defined on a Hilbert space \mathcal{H} . By the Wold Decomposition, there exists two reducing subspaces for V , say \mathcal{K} and \mathcal{L} , such that $\mathcal{H} = \mathcal{K} \oplus \mathcal{L}$ and $S = V|_{\mathcal{K}}$ is a

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE

E-mail address: wudeni.pera06@ufl.edu.

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shift operator and $U = V|_{\mathcal{L}}$ is a unitary operator. We say V is *pure*, if there is no unitary part. An isometry V is pure if and only if $\bigcap_{j=1}^{\infty} V^j(\mathcal{H}) = \{0\}$. A subspace \mathcal{W} of \mathcal{H} is called wandering subspace for V if $V^n(\mathcal{W}) \perp V^m(\mathcal{W})$ for $n \neq m$ and $\mathcal{H} = \bigoplus_{n=1}^{\infty} V^n(\mathcal{W})$. If V is a pure isometry and $\mathcal{W} = \mathcal{H} \ominus V(\mathcal{H}) = \ker(V^*)$, then $V^n(\mathcal{W}) \perp V^m(\mathcal{W})$ for $n \neq m$ and $\mathcal{H} = \bigoplus_{n=1}^{\infty} V^n(\mathcal{W})$. Hence $\ker(V^*)$ is a wandering subspace for V . Moreover, if V is a pure isometry then $V \cong M_z$ on the Hilbert Hardy space $H_{\mathcal{W}}^2$ of \mathcal{W} -valued functions for a Hilbert space \mathcal{W} with dimension of $\dim(\ker(V^*))$. The *multiplicity* of an isometry V is defined as $\text{mult}(V) = \dim(\ker(V^*))$.

A pair $V = (S, T)$ of commuting pure isometries is a *pure algebraic isopair* (or simply a *pure isopair*) if there is a nonzero polynomial $q \in \mathbb{C}[z, w]$ such that $q(S, T) = 0$. The study of pure isopairs was initiated in [AKM12]. Among the many results in that article, it is shown (see Theorem 1.20) if $V = (S, T)$ is a pure isopair, then there is a square free inner total polynomial \mathbf{p} such that $\mathbf{p}(S, T) = 0$ that is minimal in the sense if $q(S, T) = 0$, then \mathbf{p} divides q . There the notion of a *nearly cyclic* pure isopair is introduced. Here we consider nearly multi-cyclic isopairs associated with a fixed square free inner total polynomial \mathbf{p} . An isopair $V = (S, T)$ acting on a Hilbert space \mathcal{H} is called *nearly α -cyclic* if there exist $f_1, \dots, f_{\alpha} \in \mathcal{H}$ such that the closure of

$$\left\{ \sum_{j=1}^{\alpha} q_j(S, T) f_j : q_j \in \mathbb{C}[z, w] \right\}$$

is of finite codimension in \mathcal{H} and if no set of $\alpha - 1$ vectors suffices. It is *at least nearly α -cyclic* if it is not nearly γ -cyclic for any $\gamma < \alpha$.

Given a pair of isometries $V = (S, T)$, define the *bimultiplicity* of V by

$$\text{bimult}(V) = (\text{mult}(S), \text{mult}(T)).$$

The following Theorem describes a way of viewing pure isopairs as pairs of multiplication operators. The first part of the Theorem is discussed in [AKM12].

Theorem 1.1. *If $V = (S, T)$ is a pure \mathbf{p} -isopair of finite multiplicity (M, N) , then there exists an $M \times M$ matrix-valued rational inner function Φ with its poles in \mathbb{E} , such that V is unitarily equivalent to (M_z, M_{Φ}) on $H_{\mathbb{C}^M}^2$ and $\mathbf{p}(M_z, M_{\Phi}) = 0$. Moreover, letting (J_z, J_{Φ}) , denote the operators of multiplication by zI_M and Φ on $L_{\mathbb{C}^M}^2$,*

- (i) (J_z, J_{Φ}) is a commuting pair of unitary operators;
- (ii) $(J_z, J_{\Phi})|_{H_{\mathbb{C}^M}^2} = (M_z, M_{\Phi})$;
- (iii) $\mathbf{p}(\lambda, \Phi(\lambda)) = 0$ for $\lambda \in \overline{\mathbb{D}}$; and
- (iv) $\mathbf{p}(J_z, J_{\Phi}) = 0$.

Definition 1.2. *We say a point $(\lambda, \mu) \in \mathbb{C}^2$ is a regular point for \mathbf{p} if $\mathbf{p}(\lambda, \mu) = 0$ but $\nabla \mathbf{p}(\lambda, \mu) = \left(\frac{\partial \mathbf{p}}{\partial z}, \frac{\partial \mathbf{p}}{\partial w} \right) |_{(\lambda, \mu)} \neq 0$.*

Let \mathfrak{p} be a square free inner toral polynomial. Write $\mathfrak{p} = \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_s$ as a product of (distinct) irreducible factors. Then each \mathfrak{p}_j is inner toral. In other words, each $Z(\mathfrak{p}_j)$ is a distinguished variety. The zero set of \mathfrak{p} is the union of the zero sets of \mathfrak{p}_j . Let

$$\mathfrak{V}(\mathfrak{p}_j) = Z(\mathfrak{p}_j) \cap \mathbb{D}^2, \quad \mathfrak{V}(\mathfrak{p}) = Z(\mathfrak{p}) \cap \mathbb{D}^2 = \bigcup_{j=1}^s \mathfrak{V}(\mathfrak{p}_j).$$

Let \mathbb{N} denote the positive integers.

Theorem 1.3. *Suppose \mathfrak{p} is square free and inner toral and write $\mathfrak{p} = \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_s$ as a product of irreducible factors. If $V = (S, T)$ is a pure \mathfrak{p} -isopair of finite bimultiplicity (M, N) , then there exists a tuple $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}^s$ such that*

(i) *letting (n_j, m_j) denote the bidegree of \mathfrak{p}_j ,*

$$N = \sum_{j=1}^s \alpha_j n_j, \quad M = \sum_{j=1}^s \alpha_j m_j;$$

(ii) *for each j and $(\lambda, \mu) \in \mathfrak{V}(\mathfrak{p}_j)$ that is a regular point for \mathfrak{p} ,*

$$\dim(\ker(S - \lambda)^* \cap \ker(T - \mu)^*) = \alpha_j.$$

We call α the *rank* of V , denoted $\text{rank}(V)$.

Theorem 1.4. *Suppose \mathfrak{p} is square free and inner toral and write $\mathfrak{p} = \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_s$ as a product of irreducible factors. If $V = (S, T)$ is a pure \mathfrak{p} -isopair of finite bimultiplicity with rank α and $\beta = \max\{\alpha_1, \dots, \alpha_s\}$, then*

- (i) *there exists a finite codimension invariant subspace \mathcal{H} for V such that the restriction of V to \mathcal{H} is β -cyclic;*
- (ii) *V is not k -cyclic for any $k < \beta$; and*
- (iii) *there is a β -cyclic pure \mathfrak{p} -isopair V' and an invariant subspace \mathcal{K} for V' such that V is the restriction of V' to \mathcal{K} .*

Remark 1.5. Compare Theorem 1.4 with the results in [AKM12].

An important ingredient in the proof of Theorem 1.4 is a representation for a pure \mathfrak{p} isopair as a pair of multiplication operators on a reproducing kernel Hilbert space over $\mathfrak{V}(\mathfrak{p})$ in the case \mathfrak{p} is irreducible. Representations of this type already appear in the literature, [C97, Theorem D.14] for instance. Here we provide additional information. See Theorems 4.1 and 4.9.

2. PRELIMINARIES

Proposition 2.1. *Suppose $p, q \in \mathbb{C}[z, w]$.*

- (i) $Z(p) \cap Z(q)$ is a finite set if and only if p and q are relatively prime; and
- (ii) if p and q are relatively prime, then the ideal $I \subset \mathbb{C}[z, w]$ generated by p and q has finite codimension in $\mathbb{C}[z, w]$; i.e there is a finite dimensional subspace \mathcal{R} of $\mathbb{C}[z, w]$ such that for each $\psi \in \mathbb{C}[z, w]$ there exist polynomials $s, t \in \mathbb{C}[z, w]$ and $r \in \mathcal{R}$ such that

$$\psi = sp + tq + r.$$

Bezout's Theorem says that if two algebraic curves, say described by $p = 0$ and $q = 0$, do not have any common components, then they have only finitely many points in common. In particular if p and q do not have any common factors, then $Z(p)$ and $Z(q)$ have only finitely many points in common. In particular, for the ideal I generated by p and q , the affine variety $V(I) = Z(p) \cap Z(q)$ is finite. The *Finiteness Theorem* of [C97, page 13], says that if $V(I)$ is finite then the quotient ring $\mathbb{C}[z, w]/I$ has a finite dimension. Hence the ideal I has finite codimension in $\mathbb{C}[z, w]$.

For $p \in \mathbb{C}[z, w]$ and $\lambda, \mu \in \mathbb{D}$, let $p_\lambda(w) = p(\lambda, w)$ and $p^\mu(z) = p(z, \mu)$.

Lemma 2.2. *Suppose \mathbf{p} is square free and inner toral and write $\mathbf{p} = \mathbf{p}_1 \mathbf{p}_2 \cdots \mathbf{p}_s$ as a product of irreducible factors. Let q be a nonzero polynomial.*

- (i) If q vanishes on a countable subset of $\mathfrak{V}(\mathbf{p}_j)$, then \mathbf{p}_j divides q ;
- (ii) If q vanishes on a cofinite subset of $\mathfrak{V}(\mathbf{p})$, then \mathbf{p} divides q ;
- (iii) If $Z(q) \cap Z(\mathbf{p}) \cap \mathbb{D}^2$ is finite, then q and \mathbf{p} are relatively prime;
- (iv) The polynomial $\frac{\partial \mathbf{p}}{\partial w}$ has only finitely many zeros in $\mathfrak{V}(\mathbf{p})$;
- (v) If $q \frac{\partial \mathbf{p}}{\partial w}$ is zero on a cofinite subset of $\mathfrak{V}(\mathbf{p})$, then \mathbf{p} divides q ; and
- (vi) If Λ is the set of all $\lambda \in \mathbb{D}$ for which $\mathbf{p}_\lambda(w)$, as a polynomial in w , has distinct zeros, then $\Lambda \subset \mathbb{D}$ is cofinite.

Proof. To prove item (i), Proposition 2.1 item (i) implies that q and \mathbf{p}_j have a common factor. Since \mathbf{p}_j is irreducible, the conclusion follows.

Turning to item (ii), the assumption implies for each j the polynomial q vanishes on a countable subset of $\mathfrak{V}(\mathbf{p}_j)$. Hence by (i), each \mathbf{p}_j divides q . Since the \mathbf{p}_j 's are distinct, their product, \mathbf{p} , divides q too.

If q and \mathbf{p} have a common factor, then because \mathbf{p} is inner toral, $Z(q)$ and $Z(\mathbf{p})$ have infinitely many common points in \mathbb{D}^2 , proving (iii).

Let $q = \frac{\partial \mathbf{p}}{\partial w}$ and, arguing by contradiction, suppose q has infinitely many zeros in $\mathfrak{V}(\mathbf{p})$. In this case there is a j such that q has infinitely many zeros in $\mathfrak{V}(\mathbf{p}_j)$. Hence by (i), q

vanishes on $\mathfrak{V}(\mathfrak{p}_j)$. On the other hand,

$$q = \sum_{i=1}^s \frac{\partial \mathfrak{p}_i}{\partial w} \prod_{\ell \neq i} \mathfrak{p}_\ell.$$

For a point $(z, w) \in \mathfrak{V}(\mathfrak{p}_j)$ it follows that

$$\frac{\partial \mathfrak{p}_j}{\partial w} \prod_{\ell \neq j} \mathfrak{p}_\ell = 0.$$

Therefore, either $\frac{\partial \mathfrak{p}_j}{\partial w}$ has infinitely many zeros in $\mathfrak{V}(\mathfrak{p}_j)$ or there is an ℓ such that \mathfrak{p}_ℓ has infinitely many zeros in $\mathfrak{V}(\mathfrak{p}_j)$ and thus, by part (i), \mathfrak{p}_j divides $\frac{\partial \mathfrak{p}_j}{\partial w}$ or \mathfrak{p}_j divides \mathfrak{p}_ℓ . However, the polynomial \mathfrak{p}_j does not divide $\frac{\partial \mathfrak{p}_j}{\partial w}$ since the latter has lesser degree in w than the former and \mathfrak{p}_j does not divide \mathfrak{p}_ℓ since they are distinct irreducible polynomials. Therefore, $\frac{\partial \mathfrak{p}_j}{\partial w}$ has only finitely many zeros in $\mathfrak{V}(\mathfrak{p})$.

The hypothesis of (v) implies q vanishes on a cofinite subset of $\mathfrak{V}(\mathfrak{p})$. Hence (v) follows from (ii).

To prove (vi), suppose Λ is not cofinite. Then $\mathfrak{p}_\lambda(w)$ has repeated zeros at infinitely many $\lambda \in \mathbb{D}$. Thus $\frac{\partial \mathfrak{p}}{\partial w}$ has infinitely many zeros in $\mathfrak{V}(\mathfrak{p})$, a contradiction to item (iv) and hence Λ is cofinite. \square

Proposition 2.3. *Suppose $p \in \mathbb{C}[z, w]$ is a square free polynomial and write $p = p_1 p_2 \cdots p_s$ as a product of irreducible factors $p_j \in \mathbb{C}[z, w]$. If $q \in \mathbb{C}[z, w]$, then there exists $\gamma \in \mathbb{N}^s$ and an $r \in \mathbb{C}[z, w]$ such that p_j and r are relatively prime and*

$$q = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_s^{\gamma_s} r.$$

Proof. For each irreducible factor p_j , choose the largest $\gamma_j \in \mathbb{N}$ such that $p_j^{\gamma_j} | q$. Let $r = q / p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_s^{\gamma_s}$. Then $q = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_s^{\gamma_s} r$ and by the choice of γ_j , it follows that r and p_j are relatively prime. \square

Proof of Theorem 1.1. The first part of the Theorem follows from a discussion in [AKM12, Page 4]. To show that J_Φ is unitary, view J_Φ as an operator defined on $L_{\mathbb{C}^M}^2(\mathbb{T}^2)$. Since Φ is inner, $\Phi(e^{it})$ is defined and unitary for a.e t , and hence $\Phi(e^{it})^* = \Phi(e^{it})^{-1}$ for a.e t . For $f \in L_{\mathbb{C}^M}^2(\mathbb{T}^2)$,

$$J_\Phi J_\Phi^*(f) = \Phi \Phi^{-1}(f) = f.$$

Thus, $J_\Phi J_\Phi^* = I$ and likewise $J_\Phi^* J_\Phi = I$. Therefore J_Φ is unitary. Similarly, J_z is unitary, proving item (i). By the definitions, it is immediate that $(J_z, J_\Phi)|_{H_{\mathbb{C}^M}^2} = (M_z, M_\Phi)$, proving item (ii).

To prove item (iii), note that for $\gamma \in \mathbb{C}^M$ and for the Szegő's kernel $s(z, \lambda) = s_\lambda(z) = \frac{1}{1 - z\bar{\lambda}}$, $\lambda \in \mathbb{D}$,

$$\mathfrak{p}(M_z, M_\Phi)^* s_\lambda \otimes \gamma = s_\lambda \otimes \mathfrak{p}(\lambda, \Phi(\lambda))^* \gamma = 0.$$

Therefore, $\mathfrak{p}(\lambda, \Phi(\lambda)) = 0$ for $\lambda \in \mathbb{D}$. Since Φ is analytic with no poles in $\overline{\mathbb{D}}$, it follows that $\mathfrak{p}(e^{it}, \Phi(e^{it})) = 0$. Hence, $\mathfrak{p}(J_z, J_\Phi) = 0$ and $\mathfrak{p}(\lambda, \Phi(\lambda)) = 0$ for $\lambda \in \overline{\mathbb{D}}$. \square

3. RESULTS FOR GENERAL \mathfrak{p}

In this section $\mathfrak{p} = \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_s$ is a general square free inner toral polynomial with (distinct) irreducible factors \mathfrak{p}_j . Let (n_j, m_j) be the bidegree of $\mathfrak{p}_j(z, w)$.

Proposition 3.1. *Suppose $V = (S, T)$ is a pure \mathfrak{p} -isopair of finite bimultiplicity (M, N) acting on the Hilbert space \mathcal{K} . If \mathcal{H} is a finite codimension V -invariant subspace of \mathcal{K} and W is the restriction of V to \mathcal{H} , then there exists a finite codimension subspace \mathcal{L} of \mathcal{H} such that V is unitarily equivalent to the restriction of W to \mathcal{L} .*

Remark 3.2. In the case the codimension of \mathcal{H} is one, the codimension of \mathcal{L} (in \mathcal{H}) can be chosen as $N - 1$ (or as $M - 1$). In general, the proof yields a relation between the codimensions of \mathcal{H} in \mathcal{K} and \mathcal{L} in \mathcal{H} (or in \mathcal{K}).

Remark 3.3. [AKM12, Proposition 3.6] *Any nearly cyclic pure \mathfrak{p} -isopair is unitarily equivalent to a cyclic pure \mathfrak{p} -isopair restricted to a finite codimensional invariant subspace.*

Corollary 3.4. *Suppose $V = (S, T)$ is a pure \mathfrak{p} -isopair of finite bimultiplicity (M, N) acting on the Hilbert space \mathcal{K} . If there exists a finite codimension V -invariant subspace \mathcal{H} of \mathcal{K} such that the restriction of V to \mathcal{H} is β -cyclic, then there exists a β -cyclic pure isopair W acting on a Hilbert space \mathcal{L} and a finite codimension W -invariant subspace \mathcal{F} of \mathcal{L} such that $W|_{\mathcal{F}}$ is unitarily equivalent to V .*

Proof of Theorem 3.1. Following the argument in [AKM12, Proposition 3.6], let $\mathcal{F} = \mathcal{K} \ominus \mathcal{H}$ and write, with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{F}$,

$$(1) \quad V = (S, T) = \begin{pmatrix} W = (S, T)|_{\mathcal{H}} & (X, Y) \\ 0 & (A, B) \end{pmatrix}.$$

In particular A (and likewise B) are contractions on a finite dimensional Hilbert space. Because V is pure and A is a contraction, A has spectrum in the open disc \mathbb{D} . Choose a (finite) Blaschke u such that $u(A) = 0$. Note that $u(S)$ is an isometry on \mathcal{K} and moreover the codimension of the range of $u(S)$ (equal to the dimension of the kernel of $u(S)^*$) in \mathcal{K} is

(at most) dM , where d is the degree (number of zeros) of u . Further, since

$$u(S) = \begin{pmatrix} u(S|_{\mathcal{H}}) & X' \\ 0 & u(A) = 0 \end{pmatrix},$$

the range $\mathcal{L} = u(S)\mathcal{K}$ of $u(S)$ is a subspace of \mathcal{H} of finite codimension. Since $u(S)V = Wu(S)$ it follows that \mathcal{L} is invariant for W and V is unitarily equivalent to W restricted to \mathcal{L} .

To prove the remark, note that if A is a scalar (equivalently \mathcal{H} has codimension one in \mathcal{K}), then u can be chosen a single Blaschke factor. In which case the codimension of \mathcal{L} is N in \mathcal{K} and hence $N - 1$ in \mathcal{H} . In general, if d is the degree of the Blaschke u , then the codimension of \mathcal{L} in \mathcal{K} is dN . By reversing the roles of S and T one can replace N with M , the multiplicity of the shift T . \square

Proposition 3.5. *Suppose $V = (S, T)$ is a pure \mathfrak{p} -isopair of finite bimultiplicity (M, N) modeled as (M_z, M_Φ) on $\mathcal{H}_{\mathbb{C}^M}^2$ for an $M \times M$ matrix-valued rational inner function. There exists $\alpha \in \mathbb{N}^s$ and a nonzero $c \in \mathbb{C}$ such that*

- (i) $\det(w - \Phi(z)) = c \mathfrak{p}_1^{\alpha_1} \mathfrak{p}_2^{\alpha_2} \cdots \mathfrak{p}_s^{\alpha_s}$;
- (ii) $N = \sum_{j=1}^s n_j \alpha_j$ and $M = \sum_{j=1}^s m_j \alpha_j$;
- (iii) for each λ such that \mathfrak{p}_λ has M distinct zeros, $\Phi(\lambda)$ is diagonalizable and similar to

$$\bigoplus_{j=1}^s \bigoplus_{\mu_j \in Z(\mathfrak{p}_{j,\lambda})} \mu_j I_{\alpha_j};$$

- (iv) for each j and all regular points $(\lambda, \mu) \in \mathfrak{V}(\mathfrak{p}_j)$ of \mathfrak{p} ,

$$\dim [\ker((S - \lambda)^* \cap \ker(T - \mu)^*)] = \alpha_j.$$

Proof. By Theorem 1.1, there exists an $M \times M$ matrix-valued rational inner function Φ such that $V = (S, T)$ is unitarily equivalent to (M_z, M_Φ) on $\mathcal{H}_{\mathbb{C}^M}^2$ and $\mathfrak{p}(M_z, M_\Phi) = 0$. Moreover, by Theorem (1.1) item (iii),

$$(2) \quad \mathfrak{p}_\lambda(\Phi(\lambda)) = \mathfrak{p}(\lambda, \Phi(\lambda)) = 0,$$

for all $\lambda \in \overline{\mathbb{D}}$. In particular, the spectrum, $\sigma(\Phi(\lambda))$, is a subset of $Z(\mathfrak{p}_\lambda)$.

Note that $\det(wI_m - \Phi(z))$ is a rational function whose denominator $d(z)$ (a polynomial in z alone) doesn't vanish in $\overline{\mathbb{D}}$. Let $q(z, w) = d(z) \det(wI_m - \Phi(z))$, the numerator of $\det(wI_m - \Phi(z))$. Thus, for $(\lambda, \mu) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}$, μ is in the spectrum of $\Phi(\lambda)$ if and only if $q(\lambda, \mu) = 0$. In particular, $q(z, w)$ is a polynomial whose zero set in $\mathbb{D} \times \mathbb{C}$ is the set $\{(z, w) : z \in \mathbb{D}, w \in \sigma(\Phi(z))\} \subset \mathfrak{V}(\mathfrak{p})$. By Proposition 2.3, there exists an $\alpha \in \mathbb{N}^s$ and a

polynomial r such that \mathfrak{p}_j does not divide r for each j and

$$(3) \quad d(z) \det(w - \Phi(z)) = q(z, w) = \mathfrak{p}_1^{\alpha_1}(z, w) \cdots \mathfrak{p}_s^{\alpha_s}(z, w) r(z, w).$$

Observe $Z(r) \cap [\mathbb{D} \times \mathbb{C}] \subset Z(q) \cap [\mathbb{D} \times \mathbb{C}] \subset \mathfrak{V}(\mathfrak{p})$. On the other hand, r can have only finitely many zeros in $\mathfrak{V}(\mathfrak{p})$ as otherwise r has infinitely many zeros on some $\mathfrak{V}(\mathfrak{p}_j)$ and, by Lemma 2.2(i) \mathfrak{p}_j divides r . Hence $r(z, w)$ has only finitely many zeros in $\mathbb{H} = \mathbb{D} \times \mathbb{C}$. We conclude there are only finitely many $z \in \mathbb{D}$ such that $r_z(w) = r(z, w)$ has a zero and consequently r depends on z only so that $r(z, w) = r(z)$. Thus, for $\lambda \in \mathbb{D}$, the characteristic polynomial $f_\lambda(w)$ of $\Phi(\lambda)$ satisfies

$$(4) \quad f_\lambda(w) = \det(w - \Phi(\lambda)) = c(\lambda) \mathfrak{p}_{1,\lambda}^{\alpha_1}(w) \cdots \mathfrak{p}_{s,\lambda}^{\alpha_s}(w),$$

for a constant (in w) $c(\lambda)$. The degree in w on the left hand side of (4) is M and on the right hand side of (4) is, for all but finitely many λ , equal to $\sum_{j=1}^s \alpha_j m_j$. Thus $M = \sum_{j=1}^s \alpha_j m_j$. By symmetry $N = \sum_{j=1}^s \alpha_j n_j$ too.

Let Λ be the set of all $\lambda \in \mathbb{D}$ for which \mathfrak{p}_λ has $\sum_{j=1}^s m_j$ distinct zeros. By Lemma 2.2 item (vi), $\Lambda \subseteq \mathbb{D}$ is cofinite. For $\lambda \in \Lambda$, the polynomial \mathfrak{p}_λ has distinct zeros and by (2), $\mathfrak{p}_\lambda(\Phi(\lambda)) = 0$. Hence, $\Phi(\lambda)$ is diagonalizable and, for given $\mu_j \in Z(\mathfrak{p}_{j,\lambda})$, the dimension of the eigenspace $\ker(\Phi(\lambda) - \mu_j)$ is α_j . Thus $\Phi(\lambda)$ is similar to

$$\bigoplus_{j=1}^s \bigoplus_{\mu_j \in Z(\mathfrak{p}_{j,\lambda})} \mu_j I_{\alpha_j}.$$

Let $(\lambda, \mu) \in Z(\mathfrak{p}_j)$ be a regular point for \mathfrak{p} . Assume, without loss of generality, that $\frac{\partial \mathfrak{p}}{\partial w}|_{(\lambda, \mu)} \neq 0$. The minimal polynomial for $\Phi(\lambda)$ has a zero of multiplicity 1 at μ , since it divides \mathfrak{p}_λ . Hence $\Phi(\lambda)$ is similar to $\mu I_{\alpha_j} \oplus J$ where the spectrum of J does not contain μ . Therefore, the kernel of $\Phi(\lambda) - \mu$ has dimension α_j . Observe that for any $\gamma \in \ker(\Phi(\lambda) - \mu)^*$, both $(S - \lambda)^* s_\lambda \gamma = 0$ and $(T - \mu)^* s_\lambda \gamma = 0$. Hence $s_\lambda \gamma \in \ker(S - \lambda)^* \cap \ker(T - \mu)^*$. Now suppose $f \in \ker(S - \lambda)^* \cap \ker(T - \mu)^*$. Since $(S - \lambda)^* f = 0$, there is a vector $\gamma \in \mathbb{C}^N$ such that $f = s_\lambda \gamma$. Thus, $0 = (T - \mu)^* s_\lambda \gamma = s_\lambda (\Phi(\lambda)^* - \mu^*) \gamma$. Hence

$$s_\lambda \ker(\Phi(\lambda) - \mu)^* = \ker(S - \lambda)^* \cap \ker(T - \mu)^*$$

and therefore,

$$\dim [\ker(S - \lambda)^* \cap \ker(T - \mu)^*] = \dim \ker(\Phi(\lambda) - \mu)^* = \dim \ker(\Phi(\lambda) - \mu) = \alpha_j.$$

□

Proposition 3.6. *If $V = (S, T)$ is a finite bimultiplicity k -cyclic pure \mathfrak{p} -isopair acting on the Hilbert space \mathcal{K} , then for each $(\lambda, \mu) \in \mathfrak{V}(\mathfrak{p})$,*

$$\dim (\ker(S - \lambda)^* \cap \ker(T - \mu)^*) \leq k.$$

In particular, if V has rank α , then $k \geq \max\{\alpha_j : 1 \leq j \leq s\}$.

Proof. Let $\{f_1, \dots, f_k\}$ be a cyclic set for (S, T) . For any $q(z, w) \in \mathbb{C}[z, w]$, $f \in \ker(S - \lambda)^* \cap \ker(T - \mu)^*$ and $1 \leq j \leq k$,

$$\begin{aligned} \langle q(S, T)f_j, f \rangle &= \langle f_j, q(S, T)^*f \rangle \\ &= \langle f_j, q(\lambda, \mu)^*f \rangle \\ &= q(\lambda, \mu)\langle f_j, f \rangle. \end{aligned}$$

If $\dim(\ker(S - \lambda)^* \cap \ker(T - \mu)^*) > k$, then there exists a non zero vector $f \in \ker(S - \lambda)^* \cap \ker(T - \mu)^*$ perpendicular to f_j for all j . Thus $\langle q(S, T)f_j, f \rangle = 0$ for all j and for any q , and hence $\langle g, f \rangle = 0$ for any $g \in \{\sum_{j=1}^k q_j(S, T)f_j : q_j \in \mathbb{C}[z, w]\}$, a contradiction. Therefore,

$$\dim(\ker(S - \lambda)^* \cap \ker(T - \mu)^*) \leq k.$$

The last statement of the proposition follows from Proposition 3.5 item (iv). \square

Proposition 3.7. *If $V = (S, T)$ is a finite bimultiplicity pure \mathfrak{p} -isopair with rank $\alpha \in \mathbb{N}^s$ acting on a Hilbert space \mathcal{K} and if \mathcal{H} is a finite codimension V -invariant subspace of \mathcal{K} , then $W = V|_{\mathcal{H}}$ has rank α too.*

Proof. Write $W = V|_{\mathcal{H}} = (S_0, T_0)$. Let $\mathcal{F} = \mathcal{K} \ominus \mathcal{H}$. Thus \mathcal{F} has finite dimension and $\mathcal{K} = \mathcal{H} \oplus \mathcal{F}$. With respect to this decomposition, write

$$S^* = \begin{pmatrix} S_0^* & 0 \\ X^* & A^* \end{pmatrix}, \quad T^* = \begin{pmatrix} T_0^* & 0 \\ Y^* & B^* \end{pmatrix}.$$

Observe that $\sigma(A) \times \sigma(B)$ is a finite set since A and B act on a finite dimensional space. Fix $1 \leq j \leq s$. Let Γ be the set of all $(\lambda, \mu) \in \mathfrak{V}(\mathfrak{p}_j)$ such that the dimension of $\ker(S - \lambda)^* \cap \ker(T - \mu)^*$ is α_j and $(\lambda, \mu) \notin \sigma(A) \times \sigma(B)$. Hence, by Proposition 3.5 item (iv) Γ contains the cofinite set of all regular points. Since also the set $\sigma(A) \times \sigma(B)$ is finite, Γ is a cofinite subset of $\mathfrak{V}(\mathfrak{p}_j)$. Fix $(\lambda, \mu) \in \Gamma$ and let

$$\mathcal{L} = \ker(S - \lambda)^* \cap \ker(T - \mu)^* \quad \text{and} \quad \mathcal{L}_0 = \ker(S_0 - \lambda)^* \cap \ker(T_0 - \mu)^*.$$

Let $\mathcal{P} \subseteq \mathcal{H}$ be the projection of \mathcal{L} onto \mathcal{H} . Given $f \in \mathcal{L}$, write $f = f_1 \oplus f_2$, where $f_1 \in \mathcal{H}$ and $f_2 \in \mathcal{F}$. Since $f \in \mathcal{L}$, the kernel of $(S_0 - \lambda)^*$ contains f_1 . Likewise the kernel of $(T_0 - \lambda)^*$ contains f_1 . Therefore, $\mathcal{P} \subseteq \mathcal{L}_0$. If $\dim(\mathcal{L}_0) < \alpha_j$, then, since $\dim(\mathcal{L}) = \alpha_j$, there exists a non zero vector of the form $0 \oplus v$ in \mathcal{L} and hence $\ker(A - \lambda)^* \cap \ker(B - \mu)^*$ is non-empty. But, $\ker(A - \lambda)^* \cap \ker(B - \mu)^*$ is empty by the choice of (λ, μ) . Thus $\dim(\mathcal{L}_0) = \alpha_j$ for almost all (λ, μ) in $\mathfrak{V}(\mathfrak{p}_j)$. Therefore W also has rank α . \square

Corollary 3.8. *If $V = (S, T)$ is a finite bimultiplicity pure \mathfrak{p} -isopair with rank $\alpha \in \mathbb{N}^s$ acting on a Hilbert space \mathcal{K} and if \mathcal{H} is a finite codimension V -invariant subspace of \mathcal{K} , then $W = V|_{\mathcal{H}}$ is at least $\beta = \max\{\alpha_1, \dots, \alpha_s\}$ -cyclic. Hence V is at least nearly β -cyclic.*

Proof. By Proposition 3.7, W has rank α . By Proposition 3.6, W is at least β -cyclic. Thus, each restriction of V to a finite codimension invariant subspace is at least β -cyclic and hence V is at least nearly β -cyclic. \square

4. THE CASE \mathfrak{p} IS IRREDUCIBLE

In this section \mathfrak{p} is an irreducible inner toral polynomial of bidegree (n, m) .

A rank α -admissible kernel K over $\mathfrak{V}(\mathfrak{p})$ consists of a $\alpha \times m\alpha$ matrix polynomial Q and a $\alpha \times n\alpha$ matrix polynomial P such that

$$\frac{Q(z, w)Q(\zeta, \eta)^*}{1 - z\zeta^*} = K((z, w), (\zeta, \eta)) = \frac{P(z, w)P(\zeta, \eta)^*}{1 - w\eta^*}, \quad (z, w), (\zeta, \eta) \in \mathfrak{V}(\mathfrak{p})$$

where Q and P have full rank α at some point in $\mathfrak{V}(\mathfrak{p})$. In particular, at some point $x \in \mathfrak{V}(\mathfrak{p})$ the matrix $K(x, x)$ has full rank α [JKM12]. We refer to (K, P, Q) as an α -admissible triple.

Let $\mathcal{H}^2(K)$ denote the Hilbert space associated to the rank α admissible kernel K . For a point $y \in \mathfrak{V}(\mathfrak{p})$, denote by K_y the $\alpha \times \alpha$ matrix function on $\mathfrak{V}(\mathfrak{p})$ defined by $K_y(x) = K(x, y)$. Elements of $\mathcal{H}^2(K)$ are \mathbb{C}^α vector-valued functions on $\mathfrak{V}(\mathfrak{p})$ and the linear span of $\{K_y\gamma : y \in \mathfrak{V}(\mathfrak{p}), \gamma \in \mathbb{C}^\alpha\}$ is dense in $\mathcal{H}^2(K)$. Note that the operators X and Y determined densely on $\mathcal{H}^2(K)$ by $XK_{(\lambda, \mu)}\gamma = \lambda^*K_{(\lambda, \mu)}\gamma$ and $YK_{(\lambda, \mu)}\gamma = \mu^*K_{(\lambda, \mu)}\gamma$ are contractions. By Theorem (4.1) item (i) below, X^* is a bounded operator on $\mathcal{H}^2(K)$. Further for $f \in \mathcal{H}^2(K)$,

$$\langle X^*f, K_{\lambda, \mu}\gamma \rangle = \lambda \langle f(\lambda, \mu), \gamma \rangle.$$

Hence X^* is the operator of multiplication by z on $\mathcal{H}^2(K)$. Likewise, Y^* is the multiplication by w on $\mathcal{H}^2(K)$.

Theorem 4.1. *If K is a rank α -admissible kernel over $\mathfrak{V}(\mathfrak{p})$, then*

- (i) X is bounded on the linear span of $\{K_y\gamma : y \in \mathfrak{V}(\mathfrak{p}), \gamma \in \mathbb{C}^\alpha\}$;
- (ii) for each $1 \leq j \leq m\alpha$ and each positive integer n , the vector $z^n Qe_j$ (Qe_j is the j -th column of Q) lies in $\mathcal{H}^2(K)$;
- (iii) the span of $\{s_\lambda Q(\lambda, \mu)^*\gamma : (\lambda, \mu) \in \mathfrak{V}(\mathfrak{p}), \gamma \in \mathbb{C}^\alpha\}$ is dense in $\mathcal{H}_{\mathbb{C}^{m\alpha}}^2$;
- (iv) the set $\mathcal{B} = \{z^n Qe_j : n \in \mathbb{N}, 1 \leq j \leq m\alpha\}$ is an orthonormal basis for $\mathcal{H}^2(K)$; and
- (v) operators S and T densely defined on \mathcal{B} by $Sf = zf$ and $Tf = wf$ extend to a pair of pure isometries on $\mathcal{H}^2(K)$.

Proof. For a finite set of points $(\lambda_1, \mu_1), \dots, (\lambda_n, \mu_n) \in \mathfrak{V}(\mathfrak{p})$, and $\gamma_1, \dots, \gamma_n \in \mathbb{C}^\alpha$, observe that

$$\begin{aligned} \langle (I - X^*X) \sum_{j=1}^n K_{(\lambda_j, \mu_j)} \gamma_j, \sum_{k=1}^n K_{(\lambda_k, \mu_k)} \gamma_k \rangle &= \sum_{j,k=1}^n \langle (1 - \lambda_k \overline{\lambda_j}) K((\lambda_k, \mu_k), (\lambda_j, \mu_j)) \gamma_j, \gamma_k \rangle \\ &= \sum_{j,k=1}^n \langle Q(\lambda_k, \mu_k) Q^*(\lambda_j, \mu_j) \gamma_j, \gamma_k \rangle \\ &= \langle \sum_{j=1}^n Q^*(\lambda_j, \mu_j) \gamma_j, \sum_{k=1}^n Q^*(\lambda_k, \mu_k) \gamma_k \rangle \\ &\geq 0. \end{aligned}$$

Therefore, X is bounded on the linear span of $\{K_y \gamma : y \in \mathfrak{V}(\mathfrak{p}), \gamma \in \mathbb{C}^\alpha\}$.

To prove item (ii), note that by [PR16, Theorem 4.15], if f is a \mathbb{C}^α valued function defined on $\mathfrak{V}(\mathfrak{p})$ and if $K((z, w), (\zeta, \eta)) - f(z, w)f(\zeta, \eta)^*$ is a (positive semidefinite) kernel function then $f \in \mathcal{H}^2(K)$.

$$K((z, w), (\zeta, \eta)) - (z\zeta^*)^n Q(z, w)Q^*(\zeta, \eta) = \sum_{j=1}^{n-1} (z\zeta^*)^j Q(z, w)Q^*(\zeta, \eta) + (z\zeta^*)^{n+1} K((z, w), (\zeta, \eta))$$

is positive semidefinite, it follows that $z^n Qe_j \in \mathcal{H}^2(K)$.

By a result in [JKM12, Lemma 4.1], there exists a cofinite subset $\Lambda \subset \mathbb{D}$ such that for each $\lambda \in \Lambda$ there exists distinct points $\mu_1, \dots, \mu_m \in \mathbb{D}$ such that $(\lambda, \mu_j) \in \mathfrak{V}(\mathfrak{p})$ and the $m\alpha \times m\alpha$ matrix,

$$R(\lambda) := \begin{pmatrix} Q(\lambda, \mu_1)^* & \dots & Q(\lambda, \mu_m)^* \end{pmatrix}$$

has full rank. Define a map U from $\mathcal{H}^2(K)$ to $\mathcal{H}_{\mathbb{C}^{m\alpha}}^2$ by

$$UK_{(\lambda, \mu)}(z, w)\gamma = s_\lambda(z)Q(\lambda, \mu)^*\gamma.$$

Observe that for $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathbb{D}^2$ and $\gamma, \delta \in \mathbb{C}^\alpha$,

$$\begin{aligned} \langle UK_{(\lambda_1, \mu_1)}(z, w)\gamma, UK_{(\lambda_2, \mu_2)}(z, w)\delta \rangle &= \langle s_{\lambda_1}(z)Q(\lambda_2, \mu_2)Q^*(\lambda_1, \mu_1)\gamma, s_{\lambda_2}(z)\delta \rangle \\ &= \delta^* Q(\lambda_2, \mu_2)Q^*(\lambda_1, \mu_1)\gamma \langle s_{\lambda_1}(z), s_{\lambda_2}(z) \rangle \\ &= \frac{\delta^* Q(\lambda_2, \mu_2)Q^*(\lambda_1, \mu_1)\gamma}{1 - \overline{\lambda_1}\lambda_2} \\ &= \delta^* K((\lambda_2, \mu_2), (\lambda_1, \mu_1))\gamma \\ &= \langle K_{(\lambda_1, \mu_1)}(z, w)\gamma, K_{(\lambda_2, \mu_2)}(z, w)\delta \rangle. \end{aligned}$$

Therefore, U is an isometry and hence a unitary, onto its range.

Given $\lambda \in \mathbb{D}$, the span of

$$\{UK_{(\lambda, \mu_j)}\gamma : \mu_j \in Z(\mathfrak{p}_\lambda), \gamma \in \mathbb{C}^\alpha\}$$

is equal to s_λ times the span of

$$\{Q(\lambda, \mu_j)^* e_k : 1 \leq j \leq m, 1 \leq k \leq \alpha\} \subseteq \mathbb{C}^{m\alpha}.$$

If $\lambda \in \Lambda$, then $R(\lambda)$ has full rank. Thus for such λ , the span of $\{Q(\lambda, \mu)^* \gamma : \mu \text{ such that } (\lambda, \mu) \in \Gamma, \gamma \in \mathbb{C}^\alpha\}$ is all of $\mathbb{C}^{m\alpha}$. Since $\Lambda \subseteq \mathbb{D}$ is cofinite, $\{s_\lambda \mathbb{C}^{m\alpha} : \lambda \in \Lambda\}$ is dense in $\mathcal{H}_{\mathbb{C}^{m\alpha}}^2$. Since,

$$\{s_\lambda \mathbb{C}^{m\alpha} : \lambda \in \Lambda\} \subseteq \text{span}\{s_\lambda Q(\lambda, \mu)^* \gamma : (\lambda, \mu) \in \mathfrak{V}(\mathfrak{p}), \gamma \in \mathbb{C}^\alpha\},$$

the span of $\{s_\lambda Q(\lambda, \mu)^* \gamma : (\lambda, \mu) \in \mathfrak{V}(\mathfrak{p}), \gamma \in \mathbb{C}^\alpha\}$ is also dense in $\mathcal{H}_{\mathbb{C}^{m\alpha}}^2$, proving item (iii). Moreover, it proves that U is onto and hence unitary.

Let q_k denote the k -th column of Q . Thus $q_k = Qe_k$. Note that, for any $a \in \mathbb{N}$ and $1 \leq j \leq m\alpha$,

$$\begin{aligned} \langle U^* z^a e_j(\zeta, \eta), e_k \rangle &= \langle U^* z^a e_j, K_{(\zeta, \eta)} e_k \rangle \\ &= \langle z^a e_j, U K_{(\zeta, \eta)} e_k \rangle \\ &= \sum_{i=1}^{m\alpha} \langle z^a e_j, (s_\zeta q_i^*(\zeta, \eta) e_k) e_i \rangle \\ &= \langle q_j(\zeta, \eta) \zeta^a, e_k \rangle \\ &= \langle (z^a q_j)(\zeta, \eta), e_k \rangle \end{aligned}$$

and hence it follows that $U^* z^a e_j = z^a q_j$ and $U z^a q_j = z^a e_j$. In particular, $\{z^a q_j : a \in \mathbb{N}, 1 \leq j \leq m\alpha\}$ is an orthonormal basis for $\mathcal{H}^2(K)$ completing the proof of item (iv).

To prove item (v), observe that $M_z U = U S$ on \mathcal{B} and then extending to $\mathcal{H}^2(K)$, it is true on $\mathcal{H}^2(K)$ too. It is now evident that S is a pure isometry of multiplicity $m\alpha$ with wandering subspace $\{Q\gamma : \gamma \in \mathbb{C}^{m\alpha}\}$ (the span of the columns of Q). Likewise for T by symmetry. \square

Proposition 4.2 ([AM05][AM02]). *Suppose Φ is an $M \times M$ matrix-valued rational inner function and the pair (M_z, M_Φ) of multiplication operators on $\mathcal{H}_{\mathbb{C}^M}^2$ is a pure \mathfrak{p} -isopair. If the rank of the projection $I - M_\Phi M_\Phi^*$ is N , then there exists a unitary matrix U of size $(M + N) \times (M + N)$,*

$$U = \begin{pmatrix} M & N \\ A & B \\ C & D \end{pmatrix} \begin{matrix} M, \\ N \end{matrix}$$

such that

$$\Phi(z) = A + zB(I - zD)^{-1}C.$$

Proof. Since M_Φ is an isometry, $I - M_\Phi M_\Phi^*$ is a projection. Let N denote its rank. There exists an orthonormal set $\{f_1, \dots, f_N\}$ in $\mathcal{H}_{\mathbb{C}^M}^2$ such that

$$I - M_\Phi M_\Phi^* = \sum_{j=1}^N f_j f_j^*.$$

Let F denote the $N \times M$ matrix with rows f_j . Given $\lambda, \mu \in \mathbb{D}$ and $\gamma, \delta \in \mathbb{C}^M$,

$$\begin{aligned} \langle F(\mu)F^*(\lambda)\gamma, \delta \rangle &= \langle (I - M_\Phi M_\Phi^*)s_\lambda \gamma, s_\mu \delta \rangle \\ &= s(\mu, \lambda) \langle (I - \Phi(\mu)\Phi(\lambda)^*\gamma, \delta) \rangle. \end{aligned}$$

Hence

$$(5) \quad I - \Phi(\mu)\Phi(\lambda)^* = (1 - \mu\lambda^*)F(\mu)F(\lambda)^*.$$

Now do the usual lurking isometry argument. Namely, rewrite equation (5) as

$$(6) \quad I + \mu\lambda^*F(\mu)F(\lambda)^* = F(\mu)F(\lambda)^* + \Phi(\mu)\Phi(\lambda)^*.$$

Let

$$\begin{aligned} \mathcal{K} &= \text{span} \left\{ \begin{pmatrix} \gamma \\ \mu F(\mu)\gamma \end{pmatrix} : \gamma \in \mathbb{C}^M, \mu \in \mathbb{D} \right\} \subseteq \mathbb{C}^{M+N} \quad \text{and} \\ \mathcal{L} &= \text{span} \left\{ \begin{pmatrix} \Phi(\mu)\gamma \\ F(\mu)\gamma \end{pmatrix} : \gamma \in \mathbb{C}^M, \mu \in \mathbb{D} \right\} \subseteq \mathbb{C}^{M+N}. \end{aligned}$$

Define a map

$$U : \mathcal{K} \longrightarrow \mathcal{L}$$

by

$$U \begin{pmatrix} \gamma \\ \mu F(\mu)\gamma \end{pmatrix} = \begin{pmatrix} \Phi(\mu)\gamma \\ F(\mu)\gamma \end{pmatrix}.$$

By (6), U is an isometry on \mathcal{K} . If \mathcal{K} is not the whole space \mathbb{C}^{M+N} , then extend U to all of \mathbb{C}^{M+N} , so that it is a unitary. Now write U in block form as,

$$U = \begin{pmatrix} M & N \\ A & B \\ C & D \end{pmatrix} \begin{matrix} M. \\ N \end{matrix}$$

Solving the following system

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \gamma \\ \mu F(\mu)\gamma \end{pmatrix} = \begin{pmatrix} \Phi(\mu)\gamma \\ F(\mu)\gamma \end{pmatrix}$$

for $\Phi(\mu)$ and then replacing μ by z ,

$$\Phi(z) = A + zB(I - zD)^{-1}C.$$

□

An $\alpha \times \alpha$ matrix-valued kernel on a set Ω has *full rank* at $\omega \in \Omega$, if $K(\omega, \omega)$ has full rank α .

Proposition 4.3. *If $V = (S, T)$ is a finite bimultiplicity (M, N) pure \mathfrak{p} -isopair of rank α , modeled as (M_z, M_Φ) on $\mathcal{H}_{\mathbb{C}^M}^2$, where Φ is an $M \times M$ matrix-valued rational inner function, then $M = m\alpha$ and*

- (i) *there exists an $\alpha \times m\alpha$ matrix polynomial Q such that $Q(z, w)$ has full rank at almost all points of $\mathfrak{V}(\mathfrak{p})$;*
- (ii) *for $(z, w) \in \mathfrak{V}(\mathfrak{p})$*

$$Q(z, w)(\Phi(z) - w) = 0;$$

- (iii) *there exists a $\alpha \times n\alpha$ matrix polynomial P such that $P(z, w)$ has full rank at almost all points of $\mathfrak{V}(\mathfrak{p})$ and an α -admissible kernel K such that*

$$\frac{Q(z, w)Q(\zeta, \eta)^*}{1 - z\zeta^*} = K((z, w), (\zeta, \eta)) = \frac{P(z, w)P(\zeta, \eta)^*}{1 - w\eta^*} \text{ on } \mathfrak{V}(\mathfrak{p}) \times \mathfrak{V}(\mathfrak{p}).$$

Remark 4.4. *The triple (K, P, Q) in Proposition 4.3 is a rank α -admissible triple.*

Proof. Applying Proposition 3.5 to irreducible \mathfrak{p} gives $M = m\alpha$. Let Λ denote the set of $\lambda \in \mathbb{D}$ such that \mathfrak{p}_λ has m distinct zeros. By Lemma (2.2) item (vi) Λ is cofinite. Let

$$\Gamma = \{(\lambda, \mu) : \lambda \in \Lambda, \mu \in Z(\mathfrak{p}_\lambda)\}.$$

By Proposition 3.5(iii), for each $(\lambda, \mu) \in \Gamma$, the matrix $\Phi(\lambda)$ is diagonalizable and $\Phi(\lambda) - \mu$ has an α dimensional kernel. Now fix $(\lambda_0, \mu_0) \in \Gamma$. Hence there exist unitary matrices Π and Π_* such that

$$\Pi_*(\Phi(\lambda_0) - \mu_0)\Pi = \begin{pmatrix} 0_\alpha & 0 \\ 0 & A \end{pmatrix},$$

where A is $(m-1)\alpha \times (m-1)\alpha$ and invertible. Let

$$\Sigma(z, w) = \Pi_*(\Phi(z) - w)\Pi.$$

For $(\lambda, \mu) \in \Gamma$, the matrix $\Sigma(z, w)$ has an α dimensional kernel. Write,

$$\Sigma(z, w) = \begin{pmatrix} E(z) - w & G(z) \\ H(z) & L(z) - w \end{pmatrix},$$

where E is $\alpha \times \alpha$ and L is of size $(m-1)\alpha \times (m-1)\alpha$. By construction $L(z) - w$ is invertible at (λ_0, μ_0) and the other entries are 0 there. In particular, $L(\lambda) - \mu$ is invertible for almost all points $(\lambda, \mu) \in \mathfrak{V}(\mathfrak{p})$. Moreover, if $L(z) - w$ is invertible, then

$$\Sigma(z, w) = \begin{pmatrix} I & G(z) \\ 0 & L(z) - w \end{pmatrix} \begin{pmatrix} \Psi(z, w) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ (L(z) - w)^{-1}H(z) & I \end{pmatrix},$$

where

$$\Psi(z, w) = E(z) - w - G(z)(L(z) - w)^{-1}H(z).$$

Thus, on the cofinite subset of $\mathfrak{V}(\mathfrak{p})$ where $L(\lambda) - \mu$ is invertible and $\Sigma(\lambda, \mu)$ has an α dimensional kernel, $\Psi(\lambda, \mu) = 0$ and moreover,

$$\begin{pmatrix} I_\alpha & -G(\lambda)(L(\lambda) - \mu)^{-1} \end{pmatrix} \Pi_*(\Phi(\lambda) - \mu) = 0.$$

Let

$$\mathcal{Q}(z, w) = \begin{pmatrix} I_\alpha & -G(z)(L(z) - w)^{-1} \end{pmatrix} \Pi_*.$$

It follows that

$$\mathcal{Q}(z, w)(\Phi(z) - w) = 0$$

for almost all points in $\mathfrak{V}(\mathfrak{p})$. After multiplying \mathcal{Q} by an appropriate scalar polynomial we obtain an $\alpha \times m\alpha$ matrix polynomial $Q(z, w)$ that has full rank at almost all points of $\mathfrak{V}(\mathfrak{p})$ and satisfies

$$Q(z, w)(\Phi(z) - w) = 0.$$

for all $(z, w) \in \mathfrak{V}(\mathfrak{p})$.

Since T has multiplicity N , the operator M_Φ also has multiplicity N and hence the projection $I - M_\Phi M_\Phi^*$ has rank N . By Theorem 4.2, there exists a unitary matrix U of size $(M + N) \times (M + N)$,

$$U = \begin{pmatrix} M & N \\ A & B \\ C & D \end{pmatrix} \begin{matrix} M, \\ N \end{matrix}$$

such that

$$\Phi(z) = A + zB(I - zD)^{-1}C.$$

Define P by $P(z, w) = Q(z, w)B(I - zD)^{-1}$ and verify, for $(z, w) \in \mathfrak{V}(\mathfrak{p})$,

$$\begin{pmatrix} Q & zP \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} wQ & P \end{pmatrix} \text{ on } \mathfrak{V}(\mathfrak{p}).$$

It follows that, for $(\zeta, \eta) \in \mathfrak{V}(\mathfrak{p})$,

$$Q(z, w)Q(\zeta, \eta)^* + z\zeta^*P(z, w)P(\zeta, \eta)^* = w\eta^*Q(z, w)Q(\zeta, \eta)^* + P(z, w)P(\zeta, \eta)^*.$$

Rearranging gives,

$$\frac{Q(z, w)Q(\zeta, \eta)^*}{1 - z\zeta^*} = K((z, w), (\zeta, \eta)) = \frac{P(z, w)P(\zeta, \eta)^*}{1 - w\eta^*} \text{ on } \mathfrak{V}(\mathfrak{p}) \times \mathfrak{V}(\mathfrak{p}).$$

Finally, if $(\zeta, \eta) \in \mathfrak{V}(\mathfrak{p})$ is such that $Q(\zeta, \eta)$ has full rank α , then $P(\zeta, \eta)P(\zeta, \eta)^*$ also has full rank α . Therefore, $P(\zeta, \eta)$ also has full rank α and hence K is a rank α -admissible kernel. \square

Theorem 4.5. *If $V = (S, T)$ is a finite bimultiplicity (M, N) pure \mathfrak{p} -isopair with rank α , then there exists a rank α -admissible triple (K, P, Q) such that V is unitarily equivalent to the operators of multiplication by z and w on $\mathcal{H}^2(K)$.*

Proof. By Proposition 1.1, (S, T) is unitarily equivalent to (M_z, M_Φ) on $\mathcal{H}_{\mathbb{C}^M}^2$, where Φ is an $M \times M$ matrix-valued rational inner function. By Proposition 4.3, there exists a rank α -admissible triple (K, P, Q) such that

$$(7) \quad Q(z, w)(\Phi(z) - w) = 0.$$

for all $(z, w) \in \mathfrak{V}(\mathfrak{p})$. Define

$$U : \mathcal{H}_{\mathbb{C}^M}^2 \rightarrow \mathcal{H}^2(K)$$

on the span of

$$\mathcal{B} = \{s_\zeta Q^*(\zeta, \eta)\gamma : (\zeta, \eta) \in \mathfrak{V}(\mathfrak{p}), \gamma \in \mathbb{C}^\alpha\} \subseteq \mathcal{H}_{\mathbb{C}^M}^2$$

by

$$Us_\zeta(z)Q^*(\zeta, \eta)\gamma = K_{(\zeta, \eta)}(z, w)\gamma.$$

For $(\zeta, \eta) \in \mathfrak{V}(\mathfrak{p})$ and $\gamma_j \in \mathbb{C}^\alpha$ for $1 \leq j \leq 2$,

$$\begin{aligned} \langle Us_{\zeta_1}(z)Q^*(\zeta_1, \eta_1)\gamma_1, Us_{\zeta_2}(z)Q^*(\zeta_2, \eta_2)\gamma_2 \rangle &= \langle K_{(\zeta_1, \eta_1)}(z, w)\gamma_1, K_{(\zeta_2, \eta_2)}(z, w)\gamma_2 \rangle \\ &= \langle K_{(\zeta_1, \eta_1)}(\zeta_2, \eta_2)\gamma_1, \gamma_2 \rangle \\ &= \langle s_{\zeta_1}(\zeta_2)Q(\zeta_2, \eta_2)Q^*(\zeta_1, \eta_1)\gamma_1, \gamma_2 \rangle \\ &= \langle s_{\zeta_1}(z)Q^*(\zeta_1, \eta_1)\gamma_1, s_{\zeta_2}(z)Q^*(\zeta_2, \eta_2)\gamma_2 \rangle. \end{aligned}$$

Hence U is an isometry. By Theorem 4.1 item (iii) the span of \mathcal{B} is dense in $\mathcal{H}_{\mathbb{C}^M}^2$. Moreover, the range of U is dense in $\mathcal{H}^2(K)$. Thus, U is a unitary. Rewrite (7) as,

$$(8) \quad w^*Q^*(z, w) = \Phi^*(z)Q^*(z, w).$$

Let \tilde{M}_z and \tilde{M}_w be the operators of multiplication by z and w on $\mathcal{H}^2(K)$ respectively. For $(\zeta, \eta) \in \mathfrak{V}(\mathfrak{p})$ and $\gamma \in \mathbb{C}^\alpha$, using (8), observe that,

$$\begin{aligned} \tilde{M}_w^*U(s_\zeta(z)Q^*(\zeta, \eta)\gamma) &= \tilde{M}_w^*(K_{(\zeta, \eta)}(z, w)\gamma) \\ &= \bar{\eta}K_{(\zeta, \eta)}(z, w)\gamma \\ &= \bar{\eta}U(s_\zeta Q^*(\zeta, \eta)\gamma) \\ &= U(s_\zeta(z)\bar{\eta}Q^*(\zeta, \eta)\gamma) \\ &= U(s_\zeta(z)\Phi(\zeta)^*Q^*(\zeta, \eta)\gamma) \\ &= UM_\Phi^*(s_\zeta(z)Q^*(\zeta, \eta)\gamma). \end{aligned}$$

Similarly,

$$\tilde{M}_z^*U(s_\zeta(z)Q^*(\zeta, \eta)\gamma) = UM_z^*(s_\zeta(z)Q^*(\zeta, \eta)\gamma).$$

Therefore, $UM_z^* = \tilde{M}_z^*U$ and $UM_\Phi^* = \tilde{M}_w^*U$ on the span of \mathcal{B} , and hence on $\mathcal{H}_{\mathbb{C}^M}^2$. Thus our original (S, T) is unitarily equivalent to $(\tilde{M}_w, \tilde{M}_w)$ on $\mathcal{H}^2(K)$. \square

Definition 4.6. If \mathcal{B} is a subspace of vector space \mathcal{X} , then the *codimension* of \mathcal{B} in \mathcal{X} is the dimension of the quotient space \mathcal{X}/\mathcal{B} .

Lemma 4.7. Suppose \mathcal{X} is a vector space (over \mathbb{C}) and \mathcal{Q} and \mathcal{B} are subspaces of \mathcal{X} . If $\mathcal{Q} \subset \mathcal{B}$ and \mathcal{Q} has finite codimension in \mathcal{X} , then \mathcal{Q} has finite codimension in \mathcal{B} .

Proof. Since \mathcal{Q} has finite codimension in \mathcal{X} and $\mathcal{B}/\mathcal{Q} \subseteq \mathcal{X}/\mathcal{Q}$, the result follows immediately. \square

Lemma 4.8. Suppose \mathcal{K} is a Hilbert space and $\mathcal{Q} \subset \mathcal{B} \subset \mathcal{K}$ are linear subspaces (thus not necessarily closed) and let $\overline{\mathcal{Q}}$ denote the closure of \mathcal{Q} . If \mathcal{Q} has finite codimension in \mathcal{B} and if \mathcal{B} is dense in \mathcal{K} , then there exists a finite dimensional subspace \mathcal{D} of \mathcal{K} such that $\mathcal{K} = \overline{\mathcal{Q}} \oplus \mathcal{D}$.

Proof. Since \mathcal{Q} has finite codimension in \mathcal{B} , the quotient space \mathcal{B}/\mathcal{Q} has finite dimension. Let $\{[p_1 + \mathcal{Q}], \dots, [p_k + \mathcal{Q}]\}$, where $p_1, \dots, p_k \in \mathcal{B} \setminus \mathcal{Q}$, be a basis for \mathcal{B}/\mathcal{Q} . Let \mathcal{P} be the span of p_1, \dots, p_k . Then $\mathcal{Q} + \mathcal{P} = \mathcal{B}$. Let $P_{\overline{\mathcal{Q}}}$ denote the projection of $\overline{\mathcal{Q}}$ onto \mathcal{K} . Since \mathcal{P} is finite dimensional, $\mathcal{D} = (I - P_{\overline{\mathcal{Q}}})\mathcal{P}$ is finite dimensional and hence closed. Given $h \in \mathcal{K}$, there exists a sequence (h_n) in \mathcal{B} that converges to h . Write $h_n = q_n + p_n$ where $q_n \in \mathcal{Q}$ and $p_n \in \mathcal{P}$. Note that

$$h_n = q_n + p_n = q_n + P_{\overline{\mathcal{Q}}}p_n + (I - P_{\overline{\mathcal{Q}}})p_n = q'_n + p'_n,$$

where $q'_n = q_n + P_{\overline{\mathcal{Q}}}p_n \in \overline{\mathcal{Q}}$ and $p'_n = (I - P_{\overline{\mathcal{Q}}})p_n \in \mathcal{D}$. Since the sum is orthogonal, both subspaces are closed, and the sum converges, q'_n and p'_n converge to some $q \in \overline{\mathcal{Q}}$ and $p \in \mathcal{D}$ respectively. Therefore $\mathcal{K} = \overline{\mathcal{Q}} \oplus \mathcal{D}$. \square

Theorem 4.9. If K is a rank α admissible kernel function defined on $\mathfrak{V}(\mathfrak{p})$ and $S = M_z, T = M_w$ are the operators of multiplication by z and w respectively on $\mathcal{H}^2(K)$, then the pair (S, T) is nearly α -cyclic.

Proof. Since K is a rank α admissible kernel, there exist matrix polynomials Q and P of size $\alpha \times m\alpha$ and $\alpha \times n\alpha$ respectively, such that

$$K((z, w), (\zeta, \eta)) = \frac{Q(z, w)Q^*(\zeta, \eta)}{1 - z\bar{\zeta}} = \frac{P(z, w)P^*(\zeta, \eta)}{1 - w\bar{\eta}}, \quad (z, w), (\zeta, \eta) \in \mathfrak{V}(\mathfrak{p})$$

and Q and P have full rank α at some point in $\mathfrak{V}(\mathfrak{p})$. Fix $(\zeta, \eta) \in \mathfrak{V}(\mathfrak{p})$ so that $Q(\zeta, \eta)$ has full rank α . By the definition of K and [JKM12, Lemma 3.3], $K((z, w), (\zeta, \eta))$ has full rank α at almost all points in $\mathfrak{V}(\mathfrak{p})$. Let

$$Q_0 = Q_0(z, w) = Q(z, w)Q^*(\zeta, \eta).$$

Then $Q_0 e_j = (1 - S\bar{\zeta})K_{(\zeta, \eta)} e_j$. By Theorem (4.1) item (ii), $Q_0 e_j$, the j^{th} column of Q_0 , is also in $\mathcal{H}^2(K)$. Let $\tilde{q} = \tilde{q}(z, w)$ be the determinant of Q_0 . Since $K((z, w), (\zeta, \eta))$ has full rank α at almost all points in $\mathfrak{V}(\mathfrak{p})$, \tilde{q} is nonzero except for finitely many points in $\mathfrak{V}(\mathfrak{p})$. Thus, \mathfrak{p} and \tilde{q} have only finitely many common zeros in $\mathfrak{V}(\mathfrak{p})$. By Lemma 2.2 item (iii), \mathfrak{p} and q are relatively prime. Let I be the ideal generated by \mathfrak{p} and \tilde{q} . By Proposition 2.1 item (ii), $\mathbb{C}[z, w]/I$ is finite dimensional. Observe that

$$\tilde{q}_j = \tilde{q} e_j = Q_0 \text{Adj}(Q_0) e_j = \sum_{k=1}^{\alpha} b_{kj} Q_0 e_k \in \mathcal{H}^2(K),$$

where b_{kj} is the (k, j) -entry of $\text{Adj}(Q_0)$. If \vec{r} is an $\alpha \times 1$ matrix polynomial with entries r_j , then

$$(9) \quad \vec{r}\tilde{q} = \sum_{j=1}^{\alpha} r_j \text{Adj}(Q_0) Q_0 e_j \in \mathcal{H}^2(K).$$

Since $\mathbb{C}[z, w]/I$ is finite dimensional, there is a finite dimensional subspace $\mathcal{S} \subseteq \mathbb{C}[z, w]$ such that

$$\{r\tilde{q} + s\mathfrak{p} + t \mid r, s \in \mathbb{C}[z, w], t \in \mathcal{S}\} = \mathbb{C}[z, w].$$

Therefore

$$\left\{ \vec{r}\tilde{q} + \vec{s}\mathfrak{p} + \vec{t} : \vec{r}, \vec{s} \text{ are vector polynomials, } \vec{t} \in \bigoplus_1^{\alpha} \mathcal{S} \right\} = \bigoplus_1^{\alpha} \mathbb{C}[z, w].$$

and hence the span \mathcal{Q} of $\{r_1 \tilde{q}_1, \dots, r_{\alpha} \tilde{q}_{\alpha} : r_1, \dots, r_{\alpha} \in \mathbb{C}[z, w]\}$ is of finite codimension in $\bigoplus_1^{\alpha} \mathbb{C}[z, w]$.

Let $\mathcal{B} = \vee\{z^n Q e_j : n \in \mathbb{N}, 1 \leq j \leq m\alpha\} \subseteq \bigoplus_1^{\infty} \mathbb{C}[z, w]$. By equation (9) $\mathcal{Q} \subset \mathcal{B}$. By Lemma 4.7, \mathcal{Q} has finite codimension in $\bigoplus_1^{\alpha} \mathbb{C}[z, w]$. Moreover, \mathcal{B} is dense in $\mathcal{H}^2(K)$ by Theorem 4.1 item (iv). Hence by Lemma 4.8, the closure of \mathcal{Q} in $\mathcal{H}^2(K)$ has finite codimension in $\mathcal{H}^2(K)$. Equivalently, the closure of $\{\sum_{j=1}^{\alpha} r_j(S, T) \tilde{q}_j : r_j \in \mathbb{C}[z, w]\}$ is of finite codimension in $\mathcal{H}^2(K)$. Thus (S, T) is α -cyclic on $\bar{\mathcal{Q}}$ and hence at most nearly α -cyclic in $\mathcal{H}^2(K)$.

Moreover, by Corollary 3.8, (S, T) has rank at most α . For $(\zeta, \eta) \in \mathfrak{V}(\mathfrak{p})$ and for $\gamma \in \mathbb{C}^{\alpha}$, note that

$$K_{(\zeta, \eta)} \gamma \in \ker(M_z - \zeta)^* \cap \ker(M_w - \eta)^*.$$

Hence, if $(\zeta, \eta) \in \mathfrak{V}(\mathfrak{p})$ is such that $K_{(\zeta, \eta)}$ has full rank α , then $\ker(M_z - \zeta)^* \cap \ker(M_w - \eta)^*$ has dimension at least α . Therefore, (S, T) has rank at least α . Thus (S, T) has rank α . By Corollary 3.8, (S, T) is at least nearly α -cyclic and hence (S, T) is nearly α -cyclic on $\mathcal{H}^2(K)$. \square

Proposition 4.10. *If $V = (S, T)$ is a finite bimultiplicity pure \mathfrak{p} -isopair of rank α acting on the Hilbert space \mathcal{K} , then there exists a finite codimension V invariant subspace \mathcal{H} of \mathcal{K} such that the restriction of V to \mathcal{H} is α -cyclic.*

Proof. Combine Theorems 4.5 and 4.9. □

5. DECOMPOSITION OF FINITE RANK ISOPAIRS

Proposition 5.1. *Suppose $p_1, p_2 \in \mathbb{C}[z, w]$ are relatively prime, but not necessarily irreducible. If $V_j = (S_j, T_j)$ are β_j -cyclic p_j -pure isopairs, then $V = V_1 \oplus V_2$ is at most nearly $\max\{\beta_1, \beta_2\}$ -cyclic.*

Proof. Let I be the ideal generated by p_1 and p_2 . By Proposition 2.1 item (ii), I has finite codimension in $\mathbb{C}[z, w]$. Hence there exists a finite dimension subspace \mathcal{R} of $\mathbb{C}[z, w]$ such that, for each $\psi \in \mathbb{C}[z, w]$, there exist $s_1, s_2 \in \mathbb{C}[z, w]$ and $r \in \mathcal{R}$ such that

$$\psi = s_1 p_1 + s_2 p_2 + r.$$

Let \mathcal{K} denote the Hilbert space that V acts upon. Let $\beta = \max\{\beta_1, \beta_2\}$ and suppose without loss of generality $\beta_1 = \beta_2 = \beta$. Choose cyclic sets $\Gamma_j = \{\gamma_{j,1}, \dots, \gamma_{j,\beta}\}$ for V_j , for $j = 1, 2$. (In the case where $\beta_1 < \beta_2$ we can set Γ_1 to be $\{\gamma_{1,1}, \dots, \gamma_{1,\beta_1}, 0, 0, \dots, 0\}$, so that this new Γ_1 has $\beta = \beta_2$ vectors.) Let $\mathcal{K}_0 = \{\psi_1(V_1)\gamma_{1,k} \oplus \psi_2(V_2)\gamma_{2,k} : 1 \leq k \leq \beta, \psi_j \in \mathbb{C}[z, w]\}$. By the hypothesis, \mathcal{K}_0 is dense in \mathcal{K} .

For given polynomials $\psi_1, \psi_2 \in \mathbb{C}[z, w]$, there exist $s_1, s_2 \in \mathbb{C}[z, w]$ and $r \in \mathcal{R}$ such that

$$\psi_1 - \psi_2 = -s_1 p_1 + s_2 p_2 + r.$$

Rearranging gives,

$$\psi_1 + s_1 p_1 = \psi_2 + s_2 p_2 + r.$$

Let $\varphi = \psi_1 + s_1 p_1$. It follows that

$$\varphi = \psi_2 + s_2 p_2 + r.$$

Consequently,

$$\begin{aligned} \varphi(V) [\gamma_{1,k} \oplus \gamma_{2,k}] &= \varphi(V_1)\gamma_{1,k} \oplus \varphi(V_2)\gamma_{2,k} \\ &= \psi_1(V_1)\gamma_{1,k} \oplus (\psi_2(V_2)\gamma_{2,k} + r(V_2)\gamma_{2,k}). \end{aligned}$$

Let \mathcal{H}_0 denote the span of $\{\varphi(V) [\gamma_{1,k} \oplus \gamma_{2,k}] : 1 \leq k \leq \beta, \psi \in \mathbb{C}[z, w]\}$ and \mathcal{H} be the closure of \mathcal{H}_0 . Let \mathcal{L} denote the span of $\{0 \oplus r(V_2)\gamma_{2,k} : 1 \leq k \leq \beta, r \in \mathcal{R}\}$. Note that \mathcal{L} is finite dimensional since \mathcal{R} is and hence \mathcal{L} is closed. Moreover,

$$\mathcal{K}_0 = \mathcal{H}_0 + \mathcal{L}.$$

Hence \mathcal{H}_0 has finite codimension in \mathcal{K}_0 . By Lemma 4.8, \mathcal{H} has finite codimension in \mathcal{K} . Evidently \mathcal{H} is V invariant and the restriction of V to \mathcal{H} is at most β -cyclic. Therefore, V is at most nearly β -cyclic. \square

Proposition 5.2. *If $V_j = (S_j, T_j)$ are finite bimultiplicity pure \mathfrak{p}_j -isopairs with rank α_j acting on Hilbert spaces \mathcal{K}_j , for $1 \leq j \leq s$, then $\bigoplus_{j=1}^s V_j$ is nearly $\max\{\alpha_1, \alpha_2, \dots, \alpha_s\}$ -cyclic on $\bigoplus_{j=1}^s \mathcal{K}_j$.*

Proof. First suppose $s = 2$. By Proposition 4.10, each V_j is α_j -cyclic on some finite codimensional invariant subspace \mathcal{H}_j of \mathcal{K}_j . By Proposition 5.1, $V_1|_{\mathcal{H}_1} \oplus V_2|_{\mathcal{H}_2}$ is at most nearly $\max\{\alpha_1, \alpha_2\}$ -cyclic on $\mathcal{H}_1 \oplus \mathcal{H}_2$. Since each \mathcal{H}_j has finite codimension in \mathcal{K}_j , it follows that $V = V_1 \oplus V_2$ is at most nearly $\max\{\alpha_1, \alpha_2\}$ -cyclic on $\mathcal{K}_1 \oplus \mathcal{K}_2$. On the other hand, V has rank (α_1, α_2) and hence, by Corollary 3.8, is at least $\max\{\alpha_1, \alpha_2\}$ -cyclic. Thus V is nearly $\max\{\alpha_1, \alpha_2\}$ -cyclic.

Arguing by induction, suppose the result is true for $0 \leq j-1 < s$. Thus $V' = V_1 \oplus \dots \oplus V_{j-1}$ is nearly $\beta = \max\{\alpha_1, \alpha_2, \dots, \alpha_{j-1}\}$ -cyclic on $\mathcal{K}' = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \dots \oplus \mathcal{K}_{j-1}$. Hence there exists a finite codimensional invariant subspace \mathcal{H}' of \mathcal{K}' such that the restriction of V' to \mathcal{H}' is β -cyclic. Since V_j is a finite bimultiplicity \mathfrak{p}_j isopair with rank α_j , by Proposition 4.10, there exists a finite codimensional invariant subspace \mathcal{H}_j of \mathcal{K}_j such that $V_j|_{\mathcal{H}_j}$ is α_j -cyclic. Note that $\mathfrak{p}_1 \dots \mathfrak{p}_{j-1}$ and \mathfrak{p}_j are relatively prime. Applying Proposition 5.1 to $V'|_{\mathcal{H}'}$ and $V_j|_{\mathcal{H}_j}$, it follows that $V'|_{\mathcal{H}'} \oplus V_j|_{\mathcal{H}_j}$ is at most nearly $\gamma = \max\{\beta, \alpha_j\}$ -cyclic on $\mathcal{H}' \oplus \mathcal{H}_j$. Since \mathcal{H}' and \mathcal{H}_j have finite codimension in \mathcal{K}' and \mathcal{K}_j respectively, $W = V_1 \oplus V_2 \oplus \dots \oplus V_j$ is at most nearly γ -cyclic on $\mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \dots \oplus \mathcal{K}_j$. On the other hand, W has rank α and therefore at least nearly γ -cyclic by Corollary 3.8. Thus W is nearly $\gamma = \max\{\alpha_1, \dots, \alpha_j\}$ -cyclic. \square

Theorem 5.3. *Suppose $V = (S, T)$ is a finite bimultiplicity pure \mathfrak{p} -isopair of rank α acting on the Hilbert space \mathcal{K} . Then there exists a finite codimension subspace \mathcal{L} of \mathcal{K} such that the restriction of V to \mathcal{L} is $\beta = \max\{\alpha_1, \alpha_2, \dots, \alpha_s\}$ -cyclic.*

Proof. By [AKM12, Theorem 2.1], there is a finite codimension V -invariant subspace \mathcal{H} of \mathcal{K} and pure \mathfrak{p}_j -isopairs V_j such that

$$W = V|_{\mathcal{H}} = V_1 \oplus V_2 \oplus \dots \oplus V_s.$$

By Proposition 3.7, W has rank α . Hence V_j has rank α_j . By Proposition 5.2, there is a finite codimension invariant subspace \mathcal{L} of \mathcal{H} such that the restriction of W to \mathcal{L} is $\beta = \max\{\alpha_1, \alpha_2, \dots, \alpha_s\}$ -cyclic. Thus \mathcal{L} is a finite codimension subspace of \mathcal{K} such that $V|_{\mathcal{L}}$ is β -cyclic. Hence V is at most β -cyclic. By Corollary 3.8 $V|_{\mathcal{L}}$ is at least β -cyclic. Thus, $V|_{\mathcal{L}}$ is β -cyclic. \square

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